Influence analysis of measurement errors in satellite attitude determination based on extended Kalman filter

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ABSTRACT
The satellite attitude determination approach based on the Extended Kalman Filter (EKF) has been widely used in many real applications. However, the accuracy of this method largely depends on the fitness of measurement model. We aim to analyse the influence of measurement errors to the accuracy of EKF based attitude determination approach in this paper. The measurement errors, which are divided into structural error and nonstructural error by their influences, are analysed in principle. In the setting of the combination of star sensors and gyros, according to the property of innovation, we employ the technique of correlation test to analyse the influences of different kinds of measurement errors. Experimental results demonstrate the effectiveness of our previous analysis.

NOMENCLATURE
\( A(q) \) attitude matrix parameterised using the quaternion 
\( b \) constant drift 
\( \hat{C} \) estimated value of innovation sequence’s autocorrelation function 
\( d \) interrelated drift 
\( E[\bullet] \) expectation

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1.0 INTRODUCTION

Satellite attitude determination is an important technique for its control and communication. EKF based method has been considered as one of the most prominent approaches\(^{1,2}\). However, since EKF linearises the model and its performance largely depends on the accuracy of the
measurement model, there have been a lot of researches to address these deficits. For example, many researches have been proposed for Unscented Kalman Filter (UKF)\(^3,4,5\), which employs the deterministic sampling strategy to fulfill the unscented transformation. While the particle filter method\(^6\) computes the importance weights based on the random samples. Both UKF and particle filter do not need to linearise the model. Moreover, Crassidis \textit{et al.}\(^7\) proposed the Predictive Filter method, which could estimate the state variable and simultaneously, estimate the error in the state model online based on the minimum model-error principle. All these methods assume that the measurement model is accurately predefined. Certainly, there are also some other researches concerning the inaccurate measurement model. Representative work is Robust Filter\(^8\). Based on the upper bounds of linearisation and installation errors, it designs the filter that can minimise upper bound of estimated errors’ variance.

Considering the performances of these approaches, they are suitable to deal with different measurement errors and the related researches become one of the main topics in satellite attitude determination. Besides, the deep analysis of the influence of different kinds of measurement errors to the accuracy of attitude determination is also an important topic. In this paper, we analyse different kinds of measurement errors and their influences based on the widely used EKF based approach. First, the theoretical foundation of EKF for attitude determination is provided. Then, the analysis of influences of different measurement errors is presented. We reclassify the measurement errors as structural error and nonstructural error based on their influences. Finally, in the setting of attitude determination by the combination of star sensors and gyros, according to the property of innovation sequence’s autocorrelation function, we employ the normalised autocorrelation coefficients to measure the influence of different errors. Simulation results are presented to show the effectiveness.

One point should be highlighted here. In this paper, we will mainly conduct researches on the errors in measurement equation, i.e. the errors in star sensor’s measurement equation, including installation error of star sensor, error in measurement co-ordinate, rotational misalignment of star sensor and measurement noise that does not follow prior distribution. Since gyro’s measurements are employed in the state equation, we only consider one key influence factor of state equation, i.e., the influence of gyro’s installation error and compare it with the analysis of star sensor’s installation.

For analysing different kinds of measurement errors’ influence, we first derive the measurement equations with the errors. There is similar work, for example, J.Seo \textit{et al.} consider lever arm effect to Inertial Navigation System (INS)\(^9\), they derive measurement equations of GPS and odometer with lever arm influence. Furthermore, they consider the influence of other two system errors: transformation error and a scale factor error, give the new formulations of measurement equation, state equation and residual in EKF approach, and finally get good results. But there is more difference between this work and ours. In this paper, our focus is to analyse different kinds of measurement errors’ influence on attitude determination accuracy, and classify measurement errors into different groups according to the influence analysis, which we believe is helpful for deep understanding of widely used EKF based attitude determination method and designing new kinds of methods to restrain the measurement errors’ influence.

The paper is structured as follows. Section 2 presents some preliminary introduction of EKF based attitude determination approaches, especially for the system equations. We analyse the influence of different errors in detail and reclassify these errors based on their influences in Section 3. We also introduce how to use the correlation test technique to analyse the influence of different kinds of errors in this part. Experimental results are presented in Section 4, followed by the conclusion in Section 5.
2.0 EKF BASED ATTITUDE DETERMINATION APPROACH

EKF based attitude determination approach contains two important equations. The first is star sensor measurement equation. If we neglect small values with higher orders, the quaternion equation of star sensor measurement is shown as follows (10):

\[ I_{bzk,t} = h(q_t) = A_{bi}(q_t) l_{izk,t} = 2[I \Delta q \times] A_{bi}(q_t) l_{izk,t}, \quad k = 1,2,3 \quad \ldots (1) \]

Here, \( A_{bi}(q) \) is the transformation matrix from inertial frame to body frame. \( l_{izk,t} \) and \( l_{bzk,t} \), \( k = 1,2,3 \), are the co-ordinates of observation vectors in the inertial frame and body frame respectively.

We use \( \Delta q \), called error quaternion, to represent the small rotation from the estimated \( \hat{q} \) to the true attitude \( q \), then combine the first three independent components of error quaternion \( \Delta q \), the estimated error of interrelated drift \( \Delta d \) and constant drift \( \Delta b \) to formulate a new state variable that needs to estimate, i.e., \( X_{9 \times 1} = [\Delta q^T \Delta d^T \Delta b^T]^T \). The linearisation of Equation (1) can be formulated as follows.

\[ y_t = H_t X_t + v_t \quad \ldots (2) \]

Here \( H_t \) is the observation matrix. \( v_t \) is the measurement noise, which satisfies \( E(v_t v_j^T) = R_t \delta(t - j) \) and \( R_t \) is the covariance matrix of measurement errors.

The second is state equation, which is formulated from the kinematics equation.

\[ \dot{q} = \frac{1}{2} q \otimes \bar{\omega} \quad \ldots (3) \]

where \( \omega = [\omega_x \omega_y \omega_z]^T \) is the co-ordinate of angular rate in body frame. For brief, denote \( \bar{\omega} = [\omega^0 0]^T \) as the expanded angular rate.

By combining the gyro measurement equation and deducing the EKF state equation, we can finally derive the linear state Equation (10):

\[ \dot{X}(t) = F(t)X(t) + w(t) \quad \ldots (4) \]

where

\[ F(t) = \begin{bmatrix} -[\hat{\omega} \times] & -0 \cdot 5I_{3 \times 3} & -0 \cdot 5I_{3 \times 3} \\ 0_{3 \times 3} & -D_t & 0_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} \quad \begin{bmatrix} 0 & -\hat{\omega}_z & \hat{\omega}_y \\ -\hat{\omega}_z & 0 & -\hat{\omega}_x \\ -\hat{\omega}_y & \hat{\omega}_x & 0 \end{bmatrix} w(t) = \begin{bmatrix} \eta_g \\ \eta_d \\ \eta_b \end{bmatrix} \quad \ldots (5) \]

\( \eta_g \) is gyro’s measurement error, \( \eta_d \) is the error in gyro’s interrelated drift, \( \eta_b \) is the error in constant drift. They are independent with each other.

The discrete form of Equation (4) is:

\[ X(t + 1) = (I_{9 \times 9} + \Delta t \cdot F(t))X(t) + \Delta t \cdot w(t) = \Phi(t + 1,t)X(t) + \Delta t \cdot w(t) \quad \ldots (6) \]

The basic procedure of EKF based attitude determination approach is listed as follows.
(1) Construct the EKF state equation by Equation (4).

(2) Construct the EKF measurement equation by Equation (2).

(3) Estimate the state variable by traditional EKF method.

(4) After obtaining the estimation, \( \hat{X}_t = [\Delta \hat{q}_t^T \Delta \hat{d}_t^T \Delta \hat{b}_t^T]^T \) at time \( t \), we employ \( \hat{q}_t = q_{t-1} \otimes \Delta \hat{q}_t, \hat{d}_t = d_{t-1} + \Delta \hat{d}_t \) and \( \hat{b}_t = b_{t-1} + \Delta \hat{b}_t \) to derive the modified attitude and drift. Here \( \hat{q}_{t-1}, \hat{d}_{t-1}, \hat{b}_{t-1} \) represent the predicted values at time \( t \) by employing the estimated values at time \( t - 1 \).

(5) Initialise the filter by \( X_{t-1} = [0_{6 \times 1} \ 0_{6 \times 1} \ 0_{6 \times 1}]^T \) and predict \( q_{t-1}, b_{t-1}, d_{t-1} \) by estimated attitude and drift at time \( t \). Then, return to the step (3) and (4) to compute the estimated values at time \( t + 1 \).

Since gyro’s drift is compensated in this filter, we call it as feedback configuration filter.

### 3.0 INFLUENCE ANALYSIS OF DIFFERENT MEASUREMENT ERRORS

#### 3.1 Theoretical analysis of the influence of different measurement errors

We would like first to explain the relationships among weighted least square estimation, linear minimum variance estimation and Kalman filtering (KF), which is helpful to understand the relationship between measurement data and the EKF based estimation. Moreover, we give the description of measurement error sources on the spacecraft. Then we consider the influence of different measurement errors, e.g., constant measurement error or installation error, and reclassify the errors into two groups. Finally, we introduce a technique to analyse these influences caused by the errors.

First, define the adjustability of measurement data as the fitness among the measurement data, the constructed model and the predefined statistical property of measurement noise. It mainly relies on the accuracies of model and the reliability of prior knowledge. If the measurement model is accurate, there is no system error and the prior statistical properties of measurement noise are reliable, the measurement error is fully represented by random error that follows the prior distribution, then we believe the measurement data are of high adjustability. Considering the principle of weighted least square estimation, the adjustability of measurement data plays the major role in determining the performance of weighted least square estimation.

If we assign the weight matrix \( W \) in weighted least square approach by \( W = R^{-1} \), i.e., the inverse of variance matrix of measurement noise, the weighted least square approach is equivalent to linear minimum variance estimation\(^{(11)}\). The basic procedure of recursive weighted least square can be summarised as follows. Denote \( X_{k-1} \) as the optimal estimation based on the observations \( y_1, y_2, \ldots, y_{k-1} \). \( P_{k-1} \) is the corresponding covariance matrix. By integrating the observation \( y_k \), we can derive following formulations.

\[
\begin{align*}
\bar{y}_k &= H_k \bar{X}_k + v_k \\
\bar{X}_k &= \bar{X}_{k-1} + \eta_k
\end{align*}
\]  

where,

\[
\bar{X}_k = E[X_k | y_1, \ldots, y_{k-1}] = E[\Phi(t_k, t_{k-1})X_{k-1} | y_1, \ldots, y_{k-1}] = \Phi(t_k, t_{k-1})\hat{X}_{k-1} \\
E[v_k v_j^T] &= R \delta(k-j), E\eta_k = 0, E\eta_k^2 = 0, E[\eta_k \eta_k^T] = P_{k-1}, E[\eta_k v_k^T] = 0
\]
As what we have mentioned above, if \( W = R^{-1} \), we can also derive the minimum variance estimation by solving Equation (7). More interestingly, it is also the solution to traditional KF approach\(^{(11)}\). Thus, in essential, KF is also a kind of recursive weighted least square estimation approach. The adjustability of measurement data is also the key factor to determine the performance of KF method.

Since EKF is to employ KF on the linearised model, the performance of EKF also largely depends on measurement data’s adjustability. The optimal estimation of EKF based approach is the estimated value which could stably approximate the real value and consist with measurement data.

In the application of attitude determination, when a three-axis star sensor is mounted on a real spacecraft, its measurement frame cannot be perfectly aligned with predefined direction, resulting in constant bias errors in the attitude measurements, which is called installation error of star sensor in this paper. Similarly, the gyros also have this kind of errors. Moreover, for three star sensors, usually one of them is set as datum sensor. Due to the affect of environment, the installation directions of the other two star sensors are not aligned with the predefined direction relative to the datum sensor, which is called relative installation error or rotational misalignment of star sensor. Additionally, there may be errors in measurement co-ordinate caused by distortion of imaging system. Furthermore, random time varying errors in the attitude measurements are produced in star sensor that increases as the brightness of the reference stars decreases. This is the measurement noise whose variance is used to design the Kalman filter on the assumption that it is white and with a Gaussian distribution. However, there is often a short-period behavior in the measurement noise due to the change of its position on-orbit and heat influence. We call this error as the measurement noise that does not follow prior distribution.

In a word, there may be some constant error or installation error, etc. in the application, thus, the measurement model is changed. However, we also employ the original model, so we have induced new errors to the error items \( \nu \) and \( \eta \) in Equation (7). Additionally, if the measurement error is not Gaussian but we treat it as the Gaussian error, it will also take effects on the adjustability of measurement data. All of them result in the bias of estimation.

We now begin to analyse the influence of different errors in detail.

### 3.1.1 The installation error of star sensor

Assume that there are some uncompensated installation error, i.e., \( \delta \), of star sensor, denote \( \tilde{q} = q \otimes \delta \). Now, the measurement equation in Equation (2) becomes: \( y = h(q \otimes \delta) + \nu = h(\tilde{q}) + \nu \) and the kinematic equation is still \( \dot{q} = \frac{1}{2}q \otimes \dot{\omega} \). However, for the measurement equation, we use \( y = h(q) + \nu \) in applications. That is to say we use the right state equation and the wrong measurement equation in the EKF based approach, it is unavoidable to induce errors.

According to the new measurement formulation, since the installation error \( \delta \) is coupling with the estimated variable \( q \), i.e. \( \delta \) can not be eliminated in estimating \( q \), we need to determine whether the estimation, which is obtained by the right state equation and the wrong measurement equation, is the optimal EKF based estimation of \( \tilde{q} = q \otimes \delta \). If it is right, the following equations should hold:

\[
y = h(q \otimes \delta) + \nu = h(\tilde{q}) + \nu ; \quad \dot{\tilde{q}} = \frac{1}{2} \tilde{q} \otimes \tilde{\omega}
\]  

\[\ldots (8)\]

Here \( \dot{\tilde{q}} = \dot{q} \otimes \delta + q \otimes \dot{\delta} \). If \( \delta \) is a constant, \( \dot{\tilde{q}} = \dot{q} \otimes \delta = \frac{1}{2}q \otimes \delta \otimes \delta, \delta = [\omega^T 0]^T, \delta = [\delta^T 1]^T \).

After simple deducing, we have
Obviously, Equation (9) is not consistent with the form of second equation in Equation (8). It implies that the result derived by EKF is not the optimal estimation of \( \bar{q} \) either. In common cases, however, the second item in the right side of Equation (9) is very small and we can use \( \bar{q} = \frac{1}{2} \bar{q} \otimes \bar{\omega} \) for approximation. Therefore, if the model has uncompensated installation error, the estimation of EKF based approach is between \( q \) and \( \bar{q} \), closer to \( \bar{q} \).

Generally, this kind of error has the same form with the estimated variable, thus it couples with the state variable during the estimate process and the estimated bias would be mainly represented by the mean of the estimated error. So we consider it having a similar structure to the state variable and name it as structural error.

### 3.1.2 The installation error of gyro

Assume that the angular rate value of satellite and the direction of the satellite’s rotation axis in body frame are \( \omega \) and \( \mathbf{I} = (i_1, i_2, i_3)^T \). Angles between and the body frame’s x-axis, y-axis, z-axis are \( \alpha_1, \alpha_2, \alpha_3 \) respectively. Then,

\[
\omega_x = \omega \cos \alpha_1 = \omega i_1, \quad \omega_y = \omega \cos \alpha_2 = \omega i_2, \quad \omega_z = \omega \cos \alpha_3 = \omega i_3 \quad \ldots (10)
\]

If the installation direction of gyro in body frame is \( \mathbf{n} = (n_1, n_2, n_3)^T \) and the angle between \( \mathbf{I} \) and \( \mathbf{n} \) is \( \theta \), then \( \mathbf{I} \cdot \mathbf{n} = \cos \theta \) and the output of gyro is:

\[
\omega_s = \omega \cos \theta = \omega (\mathbf{n} \mathbf{I}) = \omega (n_1 i_1 + n_2 i_2 + n_3 i_3) = \omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 \quad \ldots (11)
\]

In real applications, assume that the installation directions of three gyros are \( \mathbf{n}_1 = (n_{11}, n_{12}, n_{13})^T \), \( \mathbf{n}_2 = (n_{21}, n_{22}, n_{23})^T \) and \( \mathbf{n}_3 = (n_{31}, n_{32}, n_{33})^T \), where \( n_{ij} \), \( i, j = 1, 2, 3 \) are cosine values of the angles among the \( i \)th gyro’s installation direction and the three axes in body frame respectively, they are the members of \( \mathbf{N}(t) \). The angular rate satisfies

\[
\begin{align*}
\omega_{x_1} &= \omega_1 n_{11} + \omega_2 n_{12} + \omega_3 n_{13} \\
\omega_{x_2} &= \omega_1 n_{21} + \omega_2 n_{22} + \omega_3 n_{23} \\
\omega_{x_3} &= \omega_1 n_{31} + \omega_2 n_{32} + \omega_3 n_{33} \\
\ldots (12)
\end{align*}
\]

Considering that there are drifts on the output of gyros, i.e., \( \omega_{g_i}(t) = \omega_{n_i} + \Delta \omega_i = \omega_{n_i}(t) + \delta(t) + d_i(t) = \eta_{g_i}(t) \) we employ the following measurement equation of gyro:

\[
\omega_{g_1}(t) = \mathbf{N}(t) \omega(t) + b(t) + d(t) + \eta(t) \\
\ldots (13)
\]

In the ideal cases, the gyros are installed on the three axes of body frame, i.e., \( \mathbf{n}_1 = (1,0,0)^T \), \( \mathbf{n}_2 = (0,1,0)^T \), \( \mathbf{n}_3 = (0,0,1)^T \). In real applications, however, the gyros are not installed on three axes of body system due to the existence of uncompensated installation error and the outside factors, \( \mathbf{N}(t) \neq I_{3\times3} \). We should consider their influences. When the uncompensated installation error is small, i.e., \( \mathbf{N}(t) \approx I_{3\times3} \), it can be compensated during the estimation of gyros’ drift and has little
influence on the accuracy of attitude determination. But when the uncompensated installation error is large, the estimation accuracy will be affected by this error as follows: the gyros’ installation errors reduce the accuracy of gyros’ output, then reduce the accuracy of prediction and finally take influence on the linearisation error of measurement model. That is to say the gyro’s installation error is represented as changing the linearisation error of measurement model. Since EKF based attitude determination approach is equivalent to recursive weighted least square estimation, the direct influence factor is the measurement error and EKF is not so sensitive to lineari-zation error under certain prediction accuracy, so the gyro’s installation error in certain range has little influence on the accuracy of EKF based method.

3.1.3 The error in measurement co-ordinate

For the error in measurement co-ordinate, we assume that there is a constant error $c$ in measurement equation, then the measurement equation is $y = Hx + c + v$. Based on the principle of linear minimum variance estimation\(^{(11)}\), the expression of the variance is not changed:

\[
P_{yy} = E[(y - Ey)(y - Ey)^T]
\]

\[
= E[(x - Ex)(Hx + c + \varepsilon - E(Hx + c + \varepsilon))^T]
\]

\[
= E[(x - Ex)(Hx + E(Hx) + \varepsilon)^T]
\]

\[
= E[(x - Ex)(x - Ex)^T]H^T = P_x H^T
\]

\[
P_{xy} = E[(y - Ey)(y - Ey)^T]
\]

\[
= E[(Hx + c + \varepsilon -EHx - c)(Hx + c + \varepsilon -EHx - c)^T]
\]

\[
= E[(Hx - EHx)(Hx - EHx)^T]
\]

\[
= HP_x H^T
\]

Meantime, the linear minimum variance estimation is:

\[
\hat{x} = Ex + P_{xy} P^{-1}_{yy}(y - Ey) = P_x H^T P^{-1}_{yy}(y - Ey)
\]

\[
= (H^T P^{-1}_{yy} H)^{-1} H^T P^{-1}_{yy} (y - Ey) = (H^T P^{-1}_{yy} H)^{-1} H^T P^{-1}_{yy} (y - c) = (H^T R^{-1} H)^{-1} H^T R^{-1} (y - c)
\]

It shows that the influence of constant error is determined by $\delta'$ that satisfies $\min \| (H^T R^{-1} H)^{-1} H^T R^{-1} \cdot \delta' - c \|_2$. In other words, the influence of constant error can be attributed to the structural error $\delta'$ in state estimation and the rest in residue. Different from installation error, its form is not structural and with the restriction of minimising the variance, the percent that can be regarded as structural error ($\delta'$) is very small. Therefore, the error in measurement co-ordinate is mostly represented in the residue.

3.1.4 The rotational misalignment of star sensor

Considering the measurement equation of star sensor’s principal axis, it has the same form as Equation (1) and Equation (2), except that it is from body frame to inertial frame, and $l_{bck}, k = 1,2,3$ are the co-ordinates of the star sensors’ principal axes in body system respectively. Commonly, $l_{bc1} = [1 \ 0 \ 0]^T$, $l_{bc2} = [1 \ 0 \ 1]^T$, and $l_{bc3} = [1 \ 0 \ 1]^T$. We take the star sensor whose principal axis’ direction overlaps with the z-axis of body frame as the datum sensor. The positions of other two star sensors may be not consistent to their predefined directions relative to the datum sensor, i.e., their directions are $l_{bc1} + \zeta_1$, $l_{bc2} + \zeta_2$. Therefore, the real measurement model is
In real applications, we do not consider this influence, i.e., we take \( \varsigma_1 = \varsigma_2 = 0 \) in above model. Since the observation is the co-ordinate of observation vector, we can regard the rotational misalignment as a special error in measurement co-ordinate, i.e., we have added \( A(q) \varsigma_k, k = 1, 2 \) in two groups of measurement. Differently, the structured installation error mentioned above is equivalent to add the item \( (A(q) (A(\delta) - I) l_{ck}, k = 1, 2, 3 \) in the three groups of measurement. Thus, if \( \varsigma_k \) can be written as the form of \( (A(\delta^{''} - I) l_{ck}, the rotational misalignment \( \varsigma_k \) becomes installment error. The influence of this rotational misalignment error on the estimation accuracy is represented by \( \delta^{''} \):

\[
\min \| (A(\delta^{''} - I) l_{c1} - (A(\delta^{''} - I) l_{c2} - (A(\delta^{''} - I) l_{c3}] - [\varsigma_1 \varsigma_2 0]^T \|_2
\]

Therefore, part of the rotational misalignment converts to structural error, the rest is in the residue. We take this case as existing relative installation error.

3.1.5 The measurement noise that does not follow prior distribution

In fact, this kind of error can be regarded as containing all the above-mentioned errors, since both the linearisation error and the inaccuracy of model can be regarded as adding other kinds of influences to white noise, which makes the white noise non-Gaussian. Therefore, here we mostly consider the setting with accurate measurement model plus non-Gaussian measurement noise. According to previous work\(^{(12)}\), the measurement noise in the measurement equation of star sensor’s principal axes can be approximately regarded as white noise, but in the application, the measurement noise may be not consistent with the predefined statistical character and EKF based method still use the noise as the prior statistical structure. Thus, we can derive this kind of error as the addition of two parts. The first is the noise that follows apriority and the other is regarded as the system bias. Generally, this kind of system bias does not have regular characteristic, and can’t be structured like installation error. It takes effect on the performance of EKF, especially the standard deviation of estimation.

In summary, the influences of different kinds of errors can be attributed to the inconstancy of traditional Kalman pattern (linear model plus white noise). We consider them as inducing system biases to the measurement data, i.e. reducing the adjustability of the measurement data. In essential, the influences of different errors are represented by working on the measurements in the left side of Equation (7), and then represented by the values of estimated variable and residue. From the view of error transformation, we can divide above mentioned errors as structural error and nonstructural error. The structural error is the error that couples with the state variable, and mostly represented by the mean value of the estimated error. The nonstructural error is the error that does not have the form couples with the state variable in the measurement model, and mainly represented by the standard deviation of the estimated error. Besides, when this kind of error is small (within certain range), it has little influence on the accuracy of EKF based method. Thus, most of the influence would be represented in the innovation and residue. Consequently, the installation error of star sensor is structural, the installation error of gyro, the measurement co-ordinate
error and the measurement noise that does not follow apriority are nonstructural. Star sensor’s rotational misalignment can be regarded as the combination of both structural and nonstructural errors.

3.2 Measurement error influence analysis using correlation test technique

In this section, we will introduce how to analyse the influences of structural and nonstructural errors. There is similar work in reference(13), which is related to the covariance analysis involving the truth filter and the engineering filter. The covariance analysis method is based on the covariance of the residue committed by a given filter. Since the square roots of this covariance matrix’s diagonal terms yield the time histories of standard deviations of errors in the estimates of the quantities of interest (for example, the attitude parameters), the filter design in which the errors are minimised is the best. This analysis method is mainly used to make rational design decision for filter and make it tuned for a given filter. However, in this paper, our purpose is to analyse the influence of different measurement errors based on the given feedback configuration engineering filter. This filter is not tuning due to the existence of measurement errors. Our focus is not to try to design a tuning filter when there are errors, but to analyse the influence of measurement errors. Therefore, we employ another analysis method in our paper.

Actually, in Ref. 13, the focus is to analyse the behaviour of residue. Here we pay more attention to the property of innovation. For the system Equations (2) and (6), the innovation sequence $\gamma_i$ is as follows:

$$\gamma_i = y_i - H\hat{X}_{i,i-1} = HX_i + v_i - H\hat{X}_{i,i-1} = H(X_i - \hat{X}_{i,i-1}) + v_i = He_i + v_i$$

So, $E\{\gamma_i, \gamma_{i-k}^T\} = HE\{e_i, e_{i-k}^T\}H^T + HE\{v_i, v_{i-k}^T\}$ for $k > 0$ \ldots \(14\)

Considering that $\hat{X}_{i,i} = \Phi\hat{X}_{i,i}, \hat{X}_{i,i} = \hat{X}_{i,i-1} + K(y_i - H\hat{X}_{i,i-1})$, we have:

$$e_i = \Phi(I - KH)e_{i-1} - \Phi K v_{i-1} + \Delta w_{i-1}$$

\ldots \(15\)

Carrying Equation(15) $k$-steps back, the formulation is:

$$e_i = [\Phi(I - KH)]^k e_{i-k} - \sum_{j=0}^{k} [\Phi(I - KH)]^{j-1} \Phi K v_{i-j} + \sum_{j=0}^{k} [\Phi(I - KH)]^{j-1} \Delta w_{i-j}$$ \ldots \(16\)

Then, $E\{e_i, e_{i-k}^T\} = [\Phi(I - KH)]^k P$, where $P$ is the steady-state error covariance matrix.

Post multiplying Equation (16) by $v_{i-k}^T$, $E\{e_i, v_{i-k}^T\} = -[\Phi(I - KH)]^{k-1} \Phi K R$, therefore,

$$E\{\gamma_i, \gamma_{i-k}^T\} = H[\Phi(I - KH)]^{k-1} \Phi [PH^T - K(HPH^T + R)], k > 0$$

$$E\{\gamma_i, \gamma_{i-k}^T\} = HPH^T + R, k = 0$$ \ldots \(17\)

So the autocorrelation function of innovation sequence $\gamma_i$ can be defined as $C_k = E[\gamma_i, \gamma_{i-k}^T]$. Based on the property of innovation in the optimal filter, if the model is accurate, the noise is white and its statistical character is known, then the innovation sequence is a white noise series when the estimation is optimal. Nevertheless, if the adjustability of the measurement is reduced (due
to the influence of different errors), the optimal estimation can not be obtained, and $\gamma_i$ is no longer a white noise series. According to the property of innovation sequence’s autocorrelation function, we employ the normalised autocorrelation coefficients to measure the influences of different errors\(^{(14)}\).

The estimate of innovation sequence’s autocorrelation function is defined as follows\(^{(14)}\):

$$ \hat{C}_k = (1/N) \sum_{i=k}^{N} \gamma_i \gamma_{i-k} $$

where $N$ is the number of estimate points. The covariance of $\hat{C}_k$ can be approximated as follows:

$$ \text{cov}([\hat{C}_k]_{ij}, [\hat{C}_l]_{pq}) = (1/N) \sum_{t=-\infty}^{\infty} ([\hat{C}_i]_{ip}[\hat{C}_{i+l-k}]_{jq} + [\hat{C}_{i+l}]_{jq}[\hat{C}_{i+k}]_{jp}) $$

where $[\hat{C}_k]_{ij}$ denotes the element in the $i$th row and the $j$th column of the matrix $\hat{C}_k$.

We can see from the Equation (17) that $C_k \rightarrow 0$ for large $k$. And it can be shown that infinite series in Equation (18) has a finite sum\(^{(14)}\), thus the covariance of $\hat{C}_k$ is proportional to $1/N$. It means that the estimate $\hat{C}_k$ are asymptotically unbiased and normal. Since $C_k = 0, k \neq 0$ for white noise, assume innovation sequence is white, then the covariance of $\hat{C}_k$ is

$$ \text{cov}([\hat{C}_k]_{ij}, [\hat{C}_l]_{pq}) = \begin{cases} 0 & k \neq l \\ (1/N)[C_0]_{ip}[C_0]_{jq} & k = l > 0 \\ (1/N)[C_0]_{ip}[C_0]_{jq} + [C_0]_{iq}[C_0]_{jp} & k = l = 0 \end{cases} \quad (19) $$

The estimates of the normalised autocorrelation coefficients can be computed by $[\hat{\rho}_k]_{ij} = [\hat{C}_k]_{ij}/([\hat{C}_0]_{ij})^{1/2}$. For white noise, we only care about the diagonal elements of $\hat{\rho}_k$. By combining Equation (19), we have $\text{var}([\hat{\rho}_k]_{ii}) = 1/N + O(1/N^2)$. Therefore, with the increase of $k$, the diagonal elements of $\hat{\rho}_k$ are also asymptotically normal. The 95% confidence limits for $[\hat{\rho}_k]_{ii}, k > 0$ are $\pm (1.96/N^{1/2})$. We can determine whether a innovation sequence has the character of white noise by following criterion: in the setting $[\hat{\rho}_k]_{ii}, k > 0$, check the number of times that they lie outside the band $\pm (1.96/N^{1/2})$. If the percent of this number is lower than 5% of the total, we can conclude that sequence $\gamma_i$ is white. For convenience, we call this percent as test ratio.

In the attitude determination settings, since the system equations are nonlinear, we must linearise them first in the EKF based approach, so even when the model is accurate and the measurement noise is consistent with apriority, the test ratio is still above 5%. Fortunately, the test ratio is stable and we will show this phenomenon in next section. Thus, when there is only linearisation error, we can employ this special test ratio in this case as a criterion. For convenience, we call this ratio as referenced ratio. More concretely, if there are some other errors and the test ratios, which are obtained by using the correlation test technique based on computing autocorrelation coefficients of the innovations in the EKF approach, are nearby this referenced ratio, we can regard that the existing errors do not affect the performance of EKF based method significantly.

Finally, we analyse the influences of structural and nonstructural errors respectively. Since the structural error is coupling with the state variable, its influence is mainly represented by the expectation of estimated error, i.e., the mean of estimated error is not zero. But it dose not heavily affect the test ratio since it has the same form with the estimated variable, when it is estimated in the state variable, it is difficult to indicate abnormality in residue or innovation sequence. For nonstructural error, however, its influence is mainly represented by the variance of estimated...
error, i.e., it affects the stability of it. This kind of influence can be represented by the innovations and the residues, so it takes great influence on the test ratio. In summary, for installation error of star sensor, it is structural and thus has little influence on the test ratio. For other nonstructural errors, e.g., error in measurement co-ordinate etc, they can be directly detected by checking the test ratio.

4.0 SIMULATION EXPERIMENTS

There are totally five groups of experiments for different purpose. The software used here is Mat Lab-based.

We would like first to show the simulation conditions. The standard deviation of gyros’ measurement error is \( \sigma_{ng} = 0.05 \text{deg/h} \), the standard deviation of the error in gyros’ interrelated drift is \( \sigma_{ng} = 0.1 \text{deg/h} \), the initialisation of gyros’ interrelated drift is \( d(0) = [0.1, 0.1, 0.1]^{T} \text{deg/h} \), interrelated constant of time is \( 1 \text{h} \), the gyros’ constant drift is \( b = [1, -1, 1]^{T} \text{deg/h} \), the standard deviation of the error in constant drift is \( \sigma_{nb} = 0.03 \text{deg/h} \) and the utilisation frequency of gyros’ data is 10Hz. The sample frequency of star sensors is 1Hz. The initial theoretical attitude is \( [\sqrt{3}/3 \sin(2\theta) \sqrt{3}/3 \sin(2\theta) \sqrt{3}/3 \sin(2\theta)]^{T} \text{rad} \), while the initial attitude in computing is \( [0, 0, 0, 1]^{T} \). The theoretical angle rate is \( \omega_{b} = 0.002 [\cos(10\omega) \cos(8\omega) \cos(5.7\omega)]^{T} \text{rad/s} \), \( \omega = 0.00107 \).

The first group of experiments is to show the influence of the star sensor’s installation error. Assume that this error is defined by a small rotated angle of the star sensor. For illustration, we have performed some experiments with different measurement random errors and different installation errors. Since the estimated accuracies of the three attitude angles and the three components of gyro’s drift are almost the same in this experiment, we only report their average values, i.e. AAA and ADCA. The results are shown in Table 1.

<table>
<thead>
<tr>
<th>Exp No</th>
<th>MRE (arc sec, 3σ)</th>
<th>SIE (arc sec)</th>
<th>SSE (arc sec)</th>
<th>MEE (arc sec)</th>
<th>EEStd (arc sec 1σ)</th>
<th>ADCA (º/h, 1σ)</th>
<th>AAA (arc sec 3σ)</th>
<th>TR (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
<td>0</td>
<td>Precise</td>
<td>2.51e-04</td>
<td>0.001</td>
<td>0.03</td>
<td>0.03</td>
<td>23.77</td>
</tr>
<tr>
<td>2</td>
<td>50</td>
<td>0</td>
<td>Precise</td>
<td>0.05</td>
<td>1.02</td>
<td>0.26</td>
<td>3.10</td>
<td>23.60</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
<td>0</td>
<td>Imprecise</td>
<td>1.59e-04</td>
<td>0.02</td>
<td>0.13</td>
<td>0.07</td>
<td>25.07</td>
</tr>
<tr>
<td>4</td>
<td>50</td>
<td>0</td>
<td>Imprecise</td>
<td>0.11</td>
<td>3.83</td>
<td>0.25</td>
<td>11.49</td>
<td>23.49</td>
</tr>
<tr>
<td>5</td>
<td>0.1</td>
<td>10</td>
<td>Imprecise</td>
<td>5.63</td>
<td>0.46</td>
<td>0.13</td>
<td>16.96</td>
<td>27.10</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>10</td>
<td>Imprecise</td>
<td>5.66</td>
<td>1.20</td>
<td>0.19</td>
<td>17.36</td>
<td>23.89</td>
</tr>
<tr>
<td>7</td>
<td>20</td>
<td>10</td>
<td>Imprecise</td>
<td>5.68</td>
<td>1.96</td>
<td>0.21</td>
<td>18.04</td>
<td>23.66</td>
</tr>
<tr>
<td>8</td>
<td>50</td>
<td>10</td>
<td>Imprecise</td>
<td>5.74</td>
<td>3.87</td>
<td>0.25</td>
<td>20.79</td>
<td>23.46</td>
</tr>
</tbody>
</table>

There are mainly three observations from Table 1:

1. Assume that the measurements of star sensor and gyro are with ideal accurate (Experiment 1), the prediction accuracy in next step is only affected by the accuracy of previous step. Experimental results show that even when the estimate accuracy is very high, the linearisation error is very small (in Experiment 1), the test ratio is still above 5% and it is nearly 24%. This means that the existence of linearisation error makes innovation sequence different from white noise. However, based on the estimated errors that are shown in Fig. 1, the estimation is still
stable. So we set the reference ratio as 24%. Experiment 2 shows that when the state equation is ideally accurate, EKF method could compress the influence of random noises effectively. Moreover, the random noises mainly affect the deviation of estimate error, take little influence on the mean of estimate errors and the test ratio.

(2) When we employ EKF based approach, the gyros’ measurement is not with ideal accuracy because of the estimated error of gyros’ drift, attitude estimation accuracy is not so high (Experiment 3 and 4), compared with Experiment 1 and 2. It means the imprecise of state equation will enlarge the linearisation error and then affect the accuracy of attitude determination. From the results in Table 1 and Fig. 2, we can see that the imprecise of state equation in EKF approach takes little influence to the mean of the estimate error, the convergence behavior and the test ratio.

(3) From Experiment 5-8, the installation error of star sensor takes great effect to the attitude determination accuracy. As seen from Fig. 3, this error mainly affects the mean of the estimated error. For test ratio, the installation error takes some influence on it, and with the decrease of installation error, the influence does not reduced, but it is often near the reference ratio (24%) and bellow 30%. This phenomenon indicates that the installation error is structural. It takes little influence to the character of innovation sequence. Moreover, the influence of random noise is mostly represented by the standard deviation of estimated error. With the increase of
random noise, the discrete range of the measurement data is augmented, under the weighted least square estimation criterion, the property of innovation sequence is more close to white noise. Thus, the increasing of random noise may reduce the test ratio a bit.

In summary, these results show that the installation error in star sensor is structural.

The second group of experiments is to show the influence of error in measurement co-ordinate. Assume that there are constant errors in measurement co-ordinates and other parameters are the same as the previous experiment. The results are shown in Table 2.

<table>
<thead>
<tr>
<th>Exp No</th>
<th>MRE (arc sec, 3σ)</th>
<th>MCE (arc sec)</th>
<th>MEE (arc sec)</th>
<th>EES (arc sec 1σ)</th>
<th>ADCA (°/h, 1σ)</th>
<th>AAA (arc sec 3σ)</th>
<th>TR (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.01</td>
<td>2.67-04</td>
<td>0.02</td>
<td>0.13</td>
<td>0.07</td>
<td>40.53</td>
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<tr>
<td>2</td>
<td>10.0</td>
<td>0.5</td>
<td>0.03</td>
<td>1.11</td>
<td>0.19</td>
<td>3.32</td>
<td>26.65</td>
</tr>
<tr>
<td>3</td>
<td>10.0</td>
<td>0.0</td>
<td>0.03</td>
<td>1.10</td>
<td>0.19</td>
<td>3.31</td>
<td>23.72</td>
</tr>
<tr>
<td>4</td>
<td>10.0</td>
<td>1.0</td>
<td>0.03</td>
<td>1.11</td>
<td>0.19</td>
<td>3.33</td>
<td>61.80</td>
</tr>
</tbody>
</table>

As seen from Table 2, compare Experiment 1 with the Experiment 3 in Table 1, and compare Experiment 2 and 3, the constant measurement error affects both the mean and standard deviation of estimated error, especially for the standard deviation. But these influences are not significant since it is absorbed by the innovation sequence. We can easily use test ratio to find this influence. More interestingly, with the decrease of the ratio between constant error and random error, the test ratio is also decreased and the innovation sequence is more close to Gaussian. When it is about 1/20, the test ratio is below 30%. It is close to the referenced ratio. Comparing the results in experiment 2 and 3, the constant measurement error takes little influence on the attitude estimation accuracy at this time.

In summary, the error in measurement co-ordinate can be regarded as nonstructural.

The third group of experiments is performed to show the influence of sensor’s rotational misalignment. Assume there is a constant misalignment, i.e., \( \alpha \). The principal axis’ directions of the two non-datum star sensors change from \([1 \ 0 \ 0]^T\) and \([0 \ 1 \ 0]^T\) to \([\cos \alpha \ 0 \ \sin \alpha]^T\) and \([0 \ \cos \alpha \ \sin \alpha]^T\). Other parameters are the same as that in the second group. Since the estimated accuracies of components of gyro’s drift and attitude angle are different at this time, we report them in Table 3.

<table>
<thead>
<tr>
<th>Exp No</th>
<th>MRE (arc sec, 3σ)</th>
<th>SRM (arc sec)</th>
<th>MEE (arc sec)</th>
<th>EES (arc sec 1σ)</th>
<th>DCA (°/h, 1σ)</th>
<th>AA (arc sec 3σ)</th>
<th>TR (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10.0</td>
<td>3.0</td>
<td>-1.35, 1.56, -0.09</td>
<td>1.09, 1.13, 1.14</td>
<td>0.18, 0.18, 0.18</td>
<td>5.19, 5.77, 3.42</td>
<td>54.87</td>
</tr>
<tr>
<td>2</td>
<td>3.0</td>
<td>3.0</td>
<td>-1.39, 1.56, -0.10</td>
<td>0.45, 0.46, 0.44</td>
<td>0.16, 0.16, 0.15</td>
<td>4.41, 4.89, 1.37</td>
<td>58.33</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
<td>3.0</td>
<td>-1.41, 1.56, -0.11</td>
<td>0.17, 0.16, 0.10</td>
<td>0.13, 0.12, 0.12</td>
<td>4.28, 4.72, 0.45</td>
<td>94.92</td>
</tr>
<tr>
<td>4</td>
<td>50.0</td>
<td>10.0</td>
<td>-4.39, 5.17, -0.33</td>
<td>3.76, 3.87, 4.03</td>
<td>0.24, 0.23, 0.25</td>
<td>17.35, 19.38, 12.13</td>
<td>38.76</td>
</tr>
<tr>
<td>5</td>
<td>10.0</td>
<td>10.0</td>
<td>-4.66, 5.21, -0.34</td>
<td>1.24, 1.25, 1.18</td>
<td>0.18, 0.18, 0.18</td>
<td>14.46, 16.08, 3.71</td>
<td>58.61</td>
</tr>
</tbody>
</table>
From Table 3, the constant misalignment error affects both the mean and standard deviation of estimated error, especially for the mean. It can be regarded as relative installation error. When the ratio between this constant error and the random error is above 1, the test ratio is above 55%. When it is smaller than 20%, this constant misalignment error cannot be detected by the test ratio obviously, but it also affects the attitude determination accuracy. We can see this by comparing Experiment 4 and 8 in Table 1 with result of Experiment 4 in Table 3.

In summary, the rotational misalignment of star sensor is the combination of both structural and non-structural error.

The forth group of experiments is to show the influence of gyro’s installation error. Assume that the installation error is $\alpha$, then the installation directions of gyros are changed from the axes of the body frame to $[\cos \alpha \sin \alpha \ 0]^T$, $[-\sin \alpha \cos \alpha \ 0]^T$, $[0 \ -\sin \alpha \cos \alpha]^T$. Other parameters are the same as that in the previous group. We list the results in Table 4.

We can see from the results that the gyro’s installation error mostly affects the estimate accuracy of gyro’s drift, then takes influence to the prediction accuracy. Comparing the results of Experiment 3 and 2 in Table 4 with Experiment 4 in Table 1 and Experiment 3 in Table 2, we can see that when the gyro’s installation error is small (smaller than 10 arc sec), the influence of this error is very small. However, with the increase of this error, the influence becomes significant. It mainly affects the standard deviation of estimated error and the test ratio is above 30%. When the gyro’s installation error reaches 25 arc sec, we can easily detect this installation error from the test ratio, even when the random error is large. Anyway, comparing the results in Table 4 with Table 1, we can see that influence of gyro’s installation error is much smaller than that of star sensor.

In summary, the gyro’s installation error is mostly nonstructural.

The final group of experiments is to show the influence of the measurement noise which does not follow the apriority. In fact, the influence of this kind error has been considered in previous four groups of experiments. They are equivalent to considering the influence to estimation caused by different inaccurate modeling error. As mentioned above, the inaccurate model can be regarded as the combination of accurate model and a system bias, which makes for the measurement error’s inconsistent with the apriority. Therefore, the previous four experiments can be taken as analysing the influence caused by the changed measurement error’s character, which is due to different inaccuracy modeling factors. In the following, we would like to show the experiment when the model is accurate and the measurement error has the form of cosine and sin, not Gaussian. The measurement error of star sensor is below 10 arc sec (3σ). Other parameters are the same as previous. The test ratio reaches 77.52%, we show the estimated errors in Fig. 4, it is obvious that this error heavily affects the standard deviation of the estimation error.

<table>
<thead>
<tr>
<th>Exp No</th>
<th>MRE (arc sec, 3σ)</th>
<th>GIE (arc sec)</th>
<th>MEE (arc sec)</th>
<th>EEStd (arc sec 3σ)</th>
<th>DCA (°/h, 1σ)</th>
<th>AA (arc sec 3σ)</th>
<th>TR (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>3</td>
<td>6e-4, 1e-4, 2e-4</td>
<td>0.02, 0.02, 0.02</td>
<td>0.13, 0.22, 0.22</td>
<td>0.07, 0.07, 0.07</td>
<td>28.07</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>10</td>
<td>0.07, 0.09, 0.02</td>
<td>1.06, 1.15, 1.15</td>
<td>0.18, 2.13, 2.15</td>
<td>3.19, 3.47, 3.48</td>
<td>24.19</td>
</tr>
<tr>
<td>3</td>
<td>50</td>
<td>10</td>
<td>0.35, 0.50, 0.06</td>
<td>3.68, 4.16, 4.22</td>
<td>0.24, 2.14, 2.14</td>
<td>11.09, 12.60, 12.67</td>
<td>24.05</td>
</tr>
<tr>
<td>4</td>
<td>50</td>
<td>15</td>
<td>0.37, 1.20, 0.11</td>
<td>3.71, 5.81, 5.24</td>
<td>0.24, 4.87, 4.88</td>
<td>11.20, 17.81, 15.73</td>
<td>33.43</td>
</tr>
<tr>
<td>5</td>
<td>100</td>
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<td>0.93, 7.34, 0.64</td>
<td>6.60, 28.2, 21.9</td>
<td>0.28, 14.3, 14.2</td>
<td>19.99, 87.56, 65.78</td>
<td>53.87</td>
</tr>
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</table>
Finally, we would like to summarise the experimental results as follows.

1. The structural error takes great effects on the mean of estimated errors. Thus, it also largely affects the attitude determination accuracy. However, the test ratio is often below 30%, near to the reference ratio and it is difficult to detect them from the test ratio.

2. The nonstructural error affects both the mean and standard deviation of the estimated error, especially for the standard deviation. The induction of this error would enlarge the test ratio significantly. The nonstructural error has much less influence on the attitude determination accuracy, especially within certain range, the influence can be neglected.

3. For different forms of measurement errors, installation error of star sensor is structural, error in measurement co-ordinate, installation error of gyro and measurement noise that does not follow the apriority are nonstructural, rotational misalignment of star sensor is the combination of both structural and nonstructural error.

5.0 CONCLUSIONS

In this paper, in the setting of the combination of star sensors and gyros, the influences of different measurement errors on attitude determination accuracy are investigated. We analyse the representation of the influences caused by different kinds of measurement errors, including the installation errors of star sensors and gyros, rotational misalignment of star sensor, the error in measurement co-ordinate and the measurement noise that does not follow the apriority. Based on above analysis, we category them as structural and nonstructural error. According to the property of innovation sequence’s autocorrelation function, we employ the correlation test technique based on computing normalised autocorrelation coefficients of the innovations to measure the influence of different kinds of errors. Plenty of experimental results are also provided to show the effectiveness.

Future works include the detailed influence analysis of actuator noise and the further analysis of the structure error and nonstructural error, including their relationship and identification. How to compensate these errors is also our future research.

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REFERENCES


